

# DUALIZABLE AND SEMI-FLAT OBJECTS IN ABSTRACT MODULE CATEGORIES

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**ABSTRACT.** In this paper, we define what means for an object in an abstract module category to be dualizable and we give a homological description of the direct limit closure of the dualizable objects. Our description recovers existing results of Govorov and Lazard, Oberst and Röhl, and Christensen and Holm. When applied to differential graded modules over a differential graded algebra, our description yields that a DG-module is semi-flat if and only if it can be obtained as a direct limit of finitely generated semi-free DG-modules. Finally, our description provides some new insight into the direct limit closure of the class of locally free sheaves of finite rank on a scheme.

## 1. INTRODUCTION

In the literature, one can find several results that describe how some kind of “flat object” in a suitable category can be obtained as a direct limit of simpler objects. Some examples are:

- (1) In 1968 Lazard [11], and independently Govorov [9] proved that over any ring, a module is flat if and only if it is a direct limit of finitely generated projective modules.
- (2) In 1970 Oberst and Röhl [14, Thm 3.2] proved that an additive functor on a small additive category is flat if and only if it is a direct limit of representable functors.
- (3) In 2014 Christensen and Holm [4] proved that over any ring, a complex of modules is semi-flat if and only if it is a direct limit of perfect complexes (= bounded complexes of finitely generated projective modules).
- (4) In 1994 Crawley-Boevey [5] proved that over certain schemes, a quasi-coherent sheaf is locally flat if and only if it is a direct limit of locally free sheaves of finite rank. In 2014 Brandenburg [2] established a related result for more general schemes.

In Section 3 we provide a categorical framework that makes it possible to study results and questions like the ones mentioned above. It is this framework that the term “abstract module categories” in the title refers to. From a suitably nice (axiomatically described) class  $\mathcal{S}$  of objects in such an abstract module category  $\mathcal{C}$ , we define a notion of semi-flatness (with respect to  $\mathcal{S}$ ). This definition depends only on an abstract tensor product, which is build into the aforementioned framework, and on a certain homological condition. We write  $\varinjlim \mathcal{S}$  for the class of objects in

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$\mathcal{C}$  that can be obtained as a direct limit of objects from  $\mathcal{S}$ . Our main result shows that under suitable assumptions,  $\varinjlim \mathcal{S}$  is precisely the class of semi-flat objects:

**Theorem 1.** *Let  $\mathcal{C}$  and  $\mathcal{S}$  be as in Setup 1. If  $M \in \mathcal{C}$  is semi-flat, then  $M \in \varinjlim \mathcal{S}$ . If  $\mathcal{C}$  and  $\mathcal{S}$  satisfy the conditions of Setup 2, then every  $M \in \varinjlim \mathcal{S}$  is semi-flat.*

The proof of this theorem is a generalization of the proof of [4, Thm. 1.1], which in turn is modelled on the proof of [11, Chap. I, Thm. 1.2]. A central new ingredient in the proof of Theorem 1 is an application of the generalized Hill Lemma by Stovicek [18, Thm 2.1].

The abstract module categories treated in Section 3 encompass more “concrete” module categories such as the category of left/right modules over a monoid (= ring object)  $A$  in a closed symmetric monoidal abelian category  $(\mathcal{C}_0, \otimes_1, 1, [-, -])$ ; see Pareigis [15]. For example, for  $A = 1$  the category of  $A$ -modules is nothing but  $\mathcal{C}_0$ . Applications of Theorem 1 to such concrete module categories are treated in Section 4. In this setting, the class  $\mathcal{S}$  could be (a suitable subset of) the *dualizable* objects in the category  $\mathcal{C}$  of  $A$ -modules. Dualizable objects in symmetric monoidal categories were defined and studied by Lewis and May in [12, III§1] and investigated further by Hovey, Palmieri, and Strickland in [10]; we extend the definition and the theory of such objects to categories of  $A$ -modules.

In the final Section 5, we specialize our setup even further. For some choices of a closed symmetric monoidal abelian category  $\mathcal{C}_0$  and of a monoid  $A \in \mathcal{C}_0$ , the category of  $A$ -modules turn out to be a well-known category in which the dualizable and the semi-flat objects admit hands-on descriptions. For example, a monoid  $A$  in the category  $\mathcal{C}_0 = \text{Ch}(\text{Ab})$  of chain complexes of abelian groups is the same as a differential graded algebra, and  $A$ -modules in the abstract sense are the same as differential graded  $A$ -modules. Furthermore, every finitely generated semi-free/semi-projective differential graded  $A$ -module, in the sense of Avramov, Foxby, and Halperin [1], is dualizable, and semi-flatness in the sense of [1] coincides with semi-flatness in the abstract sense of this paper. From these facts we get:

**Theorem 2.** *Let  $\mathcal{S}$  be the class of finitely generated semi-free/semi-projective differential graded modules over a differential graded algebra  $A$ . The direct limit closure of  $\mathcal{S}$  is precisely the class of semi-flat (or DG-flat) differential graded  $A$ -modules.*

Even this special case of Theorem 1 appears to be new. In the same vein, it follows that the results (1)–(3), mentioned in the beginning of the Introduction, are also consequences of Theorem 1. We also obtain results parallel to those in (4).

## 2. PRELIMINARIES

**2.1. Locally finitely presented categories.** We need some facts about locally finitely presented categories from Breitsprecher [3]. Let  $\mathcal{C}$  be a category. First recall:

**Definition 1.** An object  $K \in \mathcal{C}$  is called *finitely presented* if  $\mathcal{C}(K, -)$  commutes with filtered colimits. A category is called *locally finitely presented* if every element can be written as a direct limit of finitely presented objects.

**Definition 2.** A collection of objects  $\mathcal{S}$  is said to *generate*  $\mathcal{C}$  if given different maps  $f, g: A \rightarrow B$  there exists a map  $\sigma: S \rightarrow A$  with  $S \in \mathcal{S}$  such that  $f\sigma$  and  $g\sigma$  are different. If  $\mathcal{C}$  is abelian, this simply means that if  $A \rightarrow B$  is non-zero there is some  $S \rightarrow A$  with  $S \in \mathcal{S}$  such that  $S \rightarrow A \rightarrow B$  is non-zero.

**Proposition 1.** *Let  $\mathcal{C}$  be a Grothendieck category. Then*

- (1) [3, SATZ 1.5]  *$\mathcal{C}$  is locally finitely presented iff the finitely presented objects generate  $\mathcal{C}$ .*
- (2) [3, SATZ 1.11] *If  $\mathcal{S}$  is a set of finitely presented objects generating  $\mathcal{C}$  then  $N \in \mathcal{C}$  is finitely presented iff it has a presentation*

$$X_0 \longrightarrow X_1 \longrightarrow N \longrightarrow 0$$

*where  $X_0, X_1$  are finite sums of elements of  $\mathcal{S}$ .*

- (3) [3, SATZ 1.9] *The finitely presented objects are closed under extensions.*
- (4) [3, SATZ 1.12] *If  $\mathcal{C}$  is locally finitely presented, then the full subcategory of finitely presented objects is skeletally small, i.e. it is equivalent to a small category.*

**Lemma 1.** [5, (2.4)] *If a category  $\mathcal{C}$  is locally finitely presented and abelian, then it is Grothendieck.*

Next we look at some properties of the class  $\varinjlim \mathcal{S}$  of objects in  $\mathcal{C}$  that can be obtained as a direct limit of objects from  $\mathcal{S}$ .

**Lemma 2.** [5, Lemma p. 1664] *Let  $\mathcal{C}$  be a locally finitely presented category, let  $M \in \mathcal{C}$  and let  $\mathcal{S}$  be a collection of finitely presented objects. If any map from a finitely presented object to  $M$  factors through some  $S \in \mathcal{S}$ , then  $M \in \varinjlim \mathcal{S}$ . In particular  $\varinjlim \mathcal{S}$  is closed under direct limits and direct summands.*

*Remark 1.* Notice that the converse is true by definition for any  $\mathcal{S}$  and  $\mathcal{C}$ .

**2.2. Cotorsion pairs.** The theory of cotorsion pairs goes back to Salce [16] and has been intensively studied. See for instance Göbel and Trlifaj [8].

**Definition 3.** Let  $\mathcal{X}$  be a class of objects in an abelian category  $\mathcal{C}$ . We define

- $\mathcal{X}^\perp = \{Y \in \mathcal{C} \mid \forall X \in \mathcal{X}: \text{Ext}_{\mathcal{C}}^1(X, Y) = 0\}$
- ${}^\perp \mathcal{X} = \{Y \in \mathcal{C} \mid \forall X \in \mathcal{X}: \text{Ext}_{\mathcal{C}}^1(Y, X) = 0\}$

**Definition 4.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be classes of objects in an abelian category  $\mathcal{C}$ . We say  $(\mathcal{A}, \mathcal{B})$  is a *cotorsion pair*, if  $\mathcal{A}^\perp = \mathcal{B}$  and  ${}^\perp \mathcal{B} = \mathcal{A}$ . It is *complete* if every  $C \in \mathcal{C}$  has a presentation

$$0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$$

with  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  and a presentation

$$0 \rightarrow C \rightarrow B' \rightarrow A' \rightarrow 0$$

with  $A' \in \mathcal{A}$  and  $B' \in \mathcal{B}$ . In this paper, we are only concerned with the first presentation.

**Definition 5.** An  $\mathcal{S}$ -filtration of an object  $X$  in a category  $\mathcal{C}$  for a class of objects  $\mathcal{S}$  is a chain

$$0 = X_0 \subseteq \cdots \subseteq X_i \subseteq \cdots \subseteq X_\alpha = X$$

of objects in  $\mathcal{C}$  such that every  $X_{i+1}/X_i$  is in  $\mathcal{S}$ , and for every limit ordinal  $\alpha' \leq \alpha$  one has  $\text{colim}_{i < \alpha'} X_i = X_{\alpha'}$ . An object  $X$  called  $\mathcal{S}$ -filtered if it has an  $\mathcal{S}$ -filtration. If  $\alpha = \omega$  we say the filtration is countable, and if  $\alpha < \omega$  that it is finite. In the latter case we will also say that  $X$  is a finite extension of  $\mathcal{S}$ .

**Proposition 2.** *If  $\mathcal{S}$  is any generating set of objects in a Grothendieck category, then  $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$  is a complete cotorsion pair, and the objects in  ${}^\perp(\mathcal{S}^\perp)$  are precisely the direct summands of  $\mathcal{S}$ -filtered objects.*

*Proof.* See Saorín and Šťovíček [17, Exa. 2.8 and Cor. 2.15] (for the last assertion also see Šťovíček [18, Prop. 1.7]).  $\square$

When  $\mathcal{C}$  is locally finitely presented and  $\mathcal{S}$  consists of finitely presented objects and is closed under extensions, we can in fact realize any  $S \in {}^\perp(\mathcal{S}^\perp)$  as a direct limit. This generalizes the idea that an arbitrary direct sum can be realized as a direct limit of finite sums. The tool that allows us to generalize this idea is the generalized Hill Lemma. The full statement is rather technical so we just state here what we need (hence “weak version”):

**Lemma 3** (Hill Lemma – weak version). [18, Thm 2.1] *Let  $\mathcal{C}$  be a locally finitely presented Grothendieck category,  $\mathcal{S}$  be a set of finitely presented objects, and assume  $X$  has an  $\mathcal{S}$ -filtration. Given any map  $f: S \rightarrow X$  from a finitely presented object, then  $\text{Im}(f) \subseteq S' \subseteq X$  for some finite extension  $S'$  of elements of  $\mathcal{S}$ .*

We can now prove:

**Proposition 3.** *Let  $\mathcal{S}$  be a skeletally small class of finitely presented objects closed under finite extensions in a locally finitely presented Grothendieck category  $\mathcal{C}$ . Then any  $\mathcal{S}$ -filtered object is a direct limit of objects from  $\mathcal{S}$ . In particular,  ${}^\perp(\mathcal{S}^\perp) \subseteq \varinjlim \mathcal{S}$  when  $\mathcal{S}$  generates  $\mathcal{C}$ .*

*Proof.* Let  $X$  be an  $\mathcal{S}$ -filtered object. Since  $\mathcal{C}$  is locally finitely presented,  $X$  is also the direct limit of finitely presented objects  $X_i$ , hence also the direct limit of its finitely generated subobjects (images of finitely presented objects), but these are majored by  $\mathcal{S}$ -subobjects by Lemma 3, since  $\mathcal{S}$  is closed under finite extensions. The last statement follows from Proposition 2 and Lemma 2.  $\square$

### 3. ABSTRACT MODULE CATEGORIES

The aim in this section is to describe the direct limit closure of  $\mathcal{S}$  in the following setup:

*Setup 1.* Let  $\mathcal{C}_L$ ,  $\mathcal{C}_0$  and  $\mathcal{C}_R$  be locally finitely presented abelian categories, let  $\mathcal{S}_L \subseteq \mathcal{C}_L$  and  $\mathcal{S}_R \subseteq \mathcal{C}_R$  be generating and closed under extensions and let  $1 \in \mathcal{C}_0$  be finitely presented. Assume that we have a right continuous bifunctor (i.e. it preserves direct limits)

$$- \otimes -: \mathcal{C}_R \times \mathcal{C}_L \rightarrow \mathcal{C}_0$$

and a natural duality

$$(-)^*: \mathcal{S}_L \rightarrow \mathcal{S}_R$$

such that for any  $S \in \mathcal{S}_L$  we have natural isomorphisms

$$\begin{aligned} \mathcal{C}_0(1, S^* \otimes -) &\cong \mathcal{C}_L(S, -) \quad \text{and} \\ \mathcal{C}_0(1, - \otimes S) &\cong \mathcal{C}_R(S^*, -) \end{aligned}$$

which is then analogously true for any  $S \in \mathcal{S}_R$  by the duality between  $\mathcal{S}_L$  and  $\mathcal{S}_R$ . For simplicity we will often write  $\mathcal{C}$  for either  $\mathcal{C}_L$  or  $\mathcal{C}_R$  and  $\mathcal{S}$  for either  $\mathcal{S}_L$  and  $\mathcal{S}_R$  (see for example Theorem 1). Hopefully this should not cause any confusion.

*Remark 2.* Note that any  $S \in \mathcal{S}$  is finitely presented because  $1$  is finitely presented and  $\otimes$  is right continuous. By Proposition 1 the isomorphism classes of  $\mathcal{S}$  form a set. Also note that by Lemma 1, the category  $\mathcal{C}$  is in fact Grothendieck. When there are notational differences we will work with  $\mathcal{C}_L$  though everything could be done for  $\mathcal{C}_R$  instead.

**Example 1.** Some specific examples of Setup 1 to have in mind are:

- (1)  $A$  is a ring,  $\mathcal{C}_L/\mathcal{C}_R$  is the category  $A\text{-Mod}/\text{Mod-}A$  of left/right  $A$ -modules,  $\mathcal{C}_0 = \text{Ab}$  is the category of abelian groups,  $1$  is  $\mathbb{Z}$ ,  $\otimes = \otimes_A$  is the ordinary tensor product of modules,  $\mathcal{S}_L/\mathcal{S}_R$  is the category of finitely generated projective left/right  $A$ -modules, and  $(-)^*$  is the functor  $\text{Hom}_A(-, {}_A A_A)$ .
- (2)  $A$  is a ring,  $\mathcal{C}_L/\mathcal{C}_R$  is the category  $\text{Ch}(A\text{-Mod})/\text{Ch}(\text{Mod-}A)$  of chain complexes of left/right  $A$ -modules,  $\mathcal{C}_0$  is  $\text{Ch}(\text{Ab})$ ,  $1$  is  $\mathbb{Z}$  (viewed as a complex concentrated in degree zero),  $\otimes = \otimes_A$  is the total tensor product of chain complexes,  $\mathcal{S}_L/\mathcal{S}_R$  is the category of chain complexes of finitely generated projective left/right  $A$ -modules (these are often called *perfect complexes*), and  $(-)^*$  is the functor  $\text{Hom}_A(-, {}_A A_A)$ .
- (3)  $A$  is a DGA,  $\mathcal{C}_L/\mathcal{C}_R$  is the category  $A\text{-DGMod}/\text{DGMod-}A$  of left/right DG  $A$ -modules,  $\mathcal{C}_0$  is  $\text{Ch}(\text{Ab})$ ,  $1$  is  $\mathbb{Z}$  (as in (2)),  $\otimes = \otimes_A$  is the usual tensor product of DG-modules,  $\mathcal{S}_L/\mathcal{S}_R$  is the category of finitely generated semi-free left/right DG  $A$ -modules (that is, finite extensions of shifts of  $A$ ), and  $(-)^*$  is the functor  $\text{Hom}_A(-, {}_A A_A)$ .
- (4) If  $(\mathcal{C}_0, \otimes_1, 1, [-, -])$  is any closed symmetric monoidal abelian category, then one can take  $\mathcal{C}_L = \mathcal{C}_0 = \mathcal{C}_R$  and  $\otimes = \otimes_1$ . Moreover,  $\mathcal{S}_L = \mathcal{S}_R$  could be the subcategory of *dualizable* objects in  $\mathcal{C}_0$  (see 4.1) and  $(-)^* = [-, 1]$ .

These examples are all special cases of the “concrete module categories” studied in Section 4, and further in Section 5. A special case of (4) is where  $\mathcal{C}_0 = \text{QCoh}(X)$  is the category of quasi-coherent sheaves on a sufficiently nice scheme  $X$  and where  $\mathcal{S}_L = \mathcal{S}_R$  is the category of locally free sheaves of finite rank; see 5.4 for details.

- (5)  $\mathcal{X}$  is an additive category,  $\mathcal{C}_L/\mathcal{C}_R$  is the category  $[\mathcal{X}, \text{Ab}]/[\mathcal{X}^{\text{op}}, \text{Ab}]$  of covariant/contravariant additive functors from  $\mathcal{X}$  to  $\text{Ab}$ ,  $\mathcal{C}_0$  is  $\text{Ab}$ ,  $1$  is  $\mathbb{Z}$ ,  $\otimes = \otimes_{\mathcal{X}}$  is the tensor product from Oberst and Röhl [14],  $\mathcal{S}_L/\mathcal{S}_R$  is the category of representable covariant/contravariant functors, and the functor  $(-)^*$  maps  $\mathcal{X}(x, -)$  to  $\mathcal{X}(-, x)$  and vice versa ( $x \in \mathcal{X}$ ). See 5.5 for details.

Recall that to simplify notation we often write  $\mathcal{C}$  for either  $\mathcal{C}_L$  or  $\mathcal{C}_R$  and  $\mathcal{S}$  for either  $\mathcal{S}_L$  and  $\mathcal{S}_R$  (see Setup 1). In order to describe  $\varinjlim \mathcal{S}$ , we define from  $\mathcal{S}$  three new classes of objects in  $\mathcal{C}$ .

**Definition 6.** Let  $(\mathcal{P}, \mathcal{E})$  be the cotorsion pair in  $\mathcal{C}$  generated by  $\mathcal{S}$ . By the assumptions in Setup 1 and by Lemma 1 and Proposition 2, this cotorsion pair is complete.

Objects in  $\mathcal{P}$  are called *semi-projective* and objects in  $\mathcal{E}$  are called *acyclic* (with respect to  $\mathcal{S}$ ). An object  $M \in \mathcal{C}_L$  is called *(tensor-)flat* if the functor  $- \otimes M$  is exact. A functor  $F: \mathcal{C} \rightarrow \mathcal{C}_0$  *preserves acyclicity* if  $F(\mathcal{E}) \subseteq 1^\perp$ . Finally we say that an object  $M \in \mathcal{C}_L$  is *semi-flat* if  $M$  is flat and  $- \otimes M$  preserves acyclicity.

When necessary we shall use the more elaborate notation  $(\mathcal{P}_L, \mathcal{E}_L)$  for the cotorsion pair in  $\mathcal{C}_L$  generated by  $\mathcal{S}_L$  and similarly for  $(\mathcal{P}_R, \mathcal{E}_R)$ .

**Example 2.** We immediately see that if  $1 \in \mathcal{C}_0$  is projective, then semi-flat is the same as flat. This is for instance the case in  $A\text{-Mod}$  and  $[\mathcal{X}, \text{Ab}]$  (see (1) and (5) in Example 1), where every object is acyclic, and semi-projective is the same as projective. In  $\text{Ch}(A) = \text{Ch}(A\text{-Mod})$  and in  $A\text{-DGMod}$  (see (2) and (3) in Example 1) this is not the case, and the notions acyclic, semi-projective and semi-flat agree with the usual ones found in e.g. [1]. More on this and other examples after the main theorem.

We are now ready for the main lemma. The proof is modelled on [4, Thm 1.1] which is modelled on [11, Lem 1.1]. We try to use the same notation.

**Lemma 4.** *With the notation of Setup 1, let  $M \in \mathcal{C}_L$  be an object such that  $-\otimes M$  is left exact and  $\mathcal{C}_0(1, \varphi \otimes M)$  is epi whenever  $\varphi$  is epi in  $\mathcal{C}_R$  with  $\ker \varphi \in \mathcal{E}_R$ . Then  $M \in \varinjlim \mathcal{S}_L$ .*

*Proof.* By Lemma 2 we need to fill in the dashed part of the following diagram

$$\begin{array}{ccc} P & \xrightarrow{u} & M \\ & \searrow & \uparrow \\ & v & \downarrow w \\ & & L \end{array}$$

for some  $L \in \mathcal{S}_L$ , where  $u$  is given with  $P$  finitely presented. So let  $u$  be given.

By Proposition 1,  $P$  has a presentation

$$L_1 \xrightarrow{f} L_0 \xrightarrow{g} P \longrightarrow 0$$

with  $L_1, L_0 \in \mathcal{S}_L$ . We have an exact sequence

$$0 \longrightarrow K \xrightarrow{k} L_0^* \xrightarrow{f^*} L_1^*,$$

which, since  $-\otimes M$  and  $\mathcal{C}_0(1, -)$  are left exact, gives an exact sequence

$$0 \longrightarrow \mathcal{C}_0(1, K \otimes M) \xrightarrow{k_*} \mathcal{C}_L(L_0, M) \xrightarrow{f_*} \mathcal{C}_L(L_1, M)$$

where we have used  $\mathcal{C}_0(1, L_j^* \otimes M) \cong \mathcal{C}_L(L_j, M)$  for  $j = 0, 1$ .

By completeness of the cotorsion pair  $(\mathcal{P}_R, \mathcal{E}_R)$ , the object  $K$  has a presentation

$$0 \longrightarrow E \longrightarrow L' \xrightarrow{\varphi} K \longrightarrow 0$$

with  $L' \in \mathcal{P}_R$  and  $E \in \mathcal{E}_R$ . By assumption,  $\varphi_* = \mathcal{C}_0(1, \varphi \otimes M)$  is epi, so we get an exact sequence

$$\mathcal{C}_0(1, L' \otimes M) \xrightarrow{k_* \varphi_*} \mathcal{C}_L(L_0, M) \xrightarrow{f_*} \mathcal{C}_L(L_1, M).$$

Now since  $f_*(ug) = ugf = 0$ , we have some  $w': 1 \rightarrow L' \otimes M$  such that  $(k\varphi)_*(w') = ug$ . By Proposition 3 we can realize  $L'$  as a direct limit  $\varinjlim L_i^*$ , with  $L_i \in \mathcal{S}_L$ . This means that we have

$$L' \otimes M \cong (\varinjlim L_i^*) \otimes M \cong \varinjlim (L_i^* \otimes M),$$

as  $\otimes$  is right continuous. Since 1 is finitely presented,  $w'$  factors as

$$1 \xrightarrow{w} L^* \otimes M \xrightarrow{\iota \otimes M} L' \otimes M$$

for some  $L \in \mathcal{S}_L$  and  $w \in \mathcal{C}_L(L, M) \cong \mathcal{C}_0(1, L^* \otimes M)$ . By the assumed duality between  $\mathcal{S}_L$  and  $\mathcal{S}_R$  there exists  $v': L_0 \rightarrow L$  such that  $v'^* = k\varphi\iota$ . We now have the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}_L(L, M) & & & & \\ \downarrow \iota_* & \searrow v'_* & & & \\ \mathcal{C}_0(1, L' \otimes M) & \xrightarrow{k_*\varphi_*} & \mathcal{C}_L(L_0, M) & \xrightarrow{f_*} & \mathcal{C}_L(L_1, M) \end{array}$$

where  $wv' = v'_*(w) = k_*(\varphi_*(w')) = ug$ . This gives us the commutative diagram

$$\begin{array}{ccccccc} L_1 & \xrightarrow{f} & L_0 & \xrightarrow{g} & P & \longrightarrow & 0 \\ & \searrow & \downarrow v' & & \downarrow v & \searrow u & \\ & & 0 & & L & \xrightarrow{w} & M \end{array}$$

where  $v'f = 0$  since  $f^*v'^* = f^*k\varphi\iota = 0\varphi\iota = 0$ . Thus  $v'$  factors through  $g$  by some  $v$  as  $g$  is the cokernel of  $f$ . It remains to note that  $wv = u$ , as desired.  $\square$

*Remark 3.* The main difference between this proof and the proof in [4] is that all the relevant identities have been formalized instead of based on calculations with elements, in particular, the use of the generalized Hill Lemma instead of element considerations to find the right  $\mathcal{S}$ -subobject of a semi-projective object.

Lemma 4 will allow us to prove that every semi-flat object belongs to the direct limit closure of  $\mathcal{S}$  (see Theorem 1 below). For the converse statement, we need the following setup.

*Setup 2.* With the notation of Setup 1 assume further that  $1$  is  $\text{FP}_2$ , i.e.  $\text{Ext}_{\mathcal{C}_0}(1, -)$  respects direct limits, and that for any  $S \in \mathcal{S}_L$  we have that  $- \otimes S$  is exact and there are natural isomorphisms:

$$\begin{aligned} \text{Ext}_{\mathcal{C}_0}(1, S^* \otimes -) &\cong \text{Ext}_{\mathcal{C}_L}(S, -) \quad \text{and} \\ \text{Ext}_{\mathcal{C}_0}(1, - \otimes S) &\cong \text{Ext}_{\mathcal{C}_R}(S^*, -). \end{aligned}$$

By the duality between  $\mathcal{S}_L$  and  $\mathcal{S}_R$ , similar conditions hold for  $S \in \mathcal{S}_R$ . (Note that the isomorphisms above are the “Ext versions” of the isomorphisms from Setup 1.)

As in Remark 2 one sees that in the setting of Setup 2 every  $S \in \mathcal{S}$  is  $\text{FP}_2$ .

We can now link the direct limit closure to semi-flatness (from Definition 6).

**Theorem 1.** *Let  $\mathcal{C}$  and  $\mathcal{S}$  be as in Setup 1. If  $M \in \mathcal{C}$  is semi-flat, then  $M \in \varinjlim \mathcal{S}$ . Conversely, if  $\mathcal{C}$  and  $\mathcal{S}$  satisfy the conditions of Setup 2, then every  $M \in \varinjlim \mathcal{S}$  is semi-flat.*

*Proof.* To use Lemma 4, we just need to see, that if  $M \in \mathcal{C}_L$  is semi-flat, then  $\mathcal{C}_0(1, \varphi \otimes_A M)$  is epi whenever  $\varphi$  is epi and  $\ker \varphi$  is acyclic. This is clear, since if

$$0 \longrightarrow E \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$$

is exact and  $E$  is acyclic, then

$$0 \longrightarrow E \otimes M \longrightarrow A \otimes M \xrightarrow{\varphi \otimes M} B \otimes M \longrightarrow 0$$

is exact and  $\text{Ext}_{\mathcal{C}_0}(1, E \otimes M) = 0$ . But this implies that  $\mathcal{C}_0(1, \varphi \otimes M)$  is epi.

For the other direction we show that every  $S \in \mathcal{S}_L$  is semi-flat and that the class of semi-flat objects in  $\mathcal{C}_L$  is closed under direct limits. First observe that if  $E \in \mathcal{E}_R = \mathcal{S}_R^\perp$  and  $S \in \mathcal{S}_L$  then

$$\mathrm{Ext}_{\mathcal{C}_0}(1, E \otimes S) \cong \mathrm{Ext}_{\mathcal{C}_R}(S^*, E) = 0,$$

so  $-\otimes S$  preserves acyclicity, and since  $-\otimes S$  is assumed to be exact,  $S$  is semi-flat. Now if  $M_i \in \mathcal{C}_L$  is a direct system of semi-flat objects and  $M = \varinjlim M_i$ , then  $-\otimes M$  is exact as  $\otimes$  is right continuous and  $\varinjlim(-)$  is exact. It also preserves acyclicity, as

$$\mathrm{Ext}_{\mathcal{C}_0}(1, E \otimes \varinjlim M_i) \cong \mathrm{Ext}_{\mathcal{C}_0}(1, \varinjlim(E \otimes M_i)) \cong \varinjlim \mathrm{Ext}_{\mathcal{C}_0}(1, E \otimes M_i) = 0$$

as  $\otimes$  is right continuous,  $\mathrm{Ext}_{\mathcal{C}_0}(1, -)$  respects direct limits and  $\mathrm{Ext}_{\mathcal{C}_0}(1, E \otimes M_i) = 0$ . Hence  $M$  is semi-flat.  $\square$

We also have a simplified version of Setup 2:

*Setup 2'.* With the notation of Setup 1 assume further that that  $1 \in \mathcal{C}_0$  is projective and that every  $S \in \mathcal{S}$  is (tensor-)flat.

**Corollary 1.** *Let  $\mathcal{C}$  and  $\mathcal{S}$  be as in Setup 2'. Then the direct limit closure of  $\mathcal{S}$  is the class of flat objects in  $\mathcal{C}$ .*

*Proof.* As in Example 2 semi-flat just means flat if  $1$  is projective, so by Theorem 1 every flat object in  $\mathcal{C}$  is in the direct limit closure of  $\mathcal{S}$ . On the other hand any  $S \in \varinjlim \mathcal{S}$  is flat as this is preserved by direct limits as in the proof of Theorem 1.  $\square$

As mentioned in the Introduction, we will now see how Theorem 1 recovers Govorov and Lazard's original theorem for modules, the theorem by Christensen and Holm for complexes of modules, the theorem by Oberst and Röhl for functor categories, and how it gives a new result for differential graded modules. We will also look at the case of quasi-coherent sheaves.

Most of these examples are built of categories of left/right objects for some monoid in a symmetric monoidal category. So in the next section we will explain this construction with a new definition of *dualizable* objects in such categories and show in what cases they satisfy Setup 1 and 2. Then we will go in depth with the more concrete examples, calculating the different classes of objects.

#### 4. CONCRETE MODULE CATEGORIES

*Setup 3.* The details of this setup can be found in Pareigis [15]. Consider any closed symmetric monoidal abelian category  $\mathcal{C}_0 = (\mathcal{C}_0, \otimes_1, [-, -], 1)$ . A *monoid* (or a *ring object*) in  $\mathcal{C}_0$  is an object,  $A$ , together with an associative multiplication  $A \otimes_1 A \rightarrow A$  with a unit  $1 \rightarrow A$ . We can then consider the category  ${}_A\mathcal{C}$  of *left  $A$ -modules* whose objects are objects  $X \in \mathcal{C}_0$  equipped with a left  $A$ -multiplication  $A \otimes_1 X \rightarrow X$  respecting the multiplication of  $A$  on the left and the unit. The morphisms are morphisms in  $\mathcal{C}_0$  respecting this left  $A$ -multiplication. We can also consider the category  $\mathcal{C}_A$  of *right  $A$ -modules* and the category  ${}_A\mathcal{C}_A$  of  *$(A, A)$ -bimodules*, that is, simultaneously left and right  $A$ -modules with compatible actions.

We can then construct a functor  $\otimes_A: \mathcal{C}_A \times {}_A\mathcal{C} \rightarrow \mathcal{C}_0$  as a coequalizer:

$$Y \otimes_1 A \otimes_1 X \rightrightarrows Y \otimes_1 X \longrightarrow Y \otimes_A X.$$

And we get induced functors  ${}_A\mathcal{C}_A \times {}_A\mathcal{C} \rightarrow {}_A\mathcal{C}$  and  $\mathcal{C}_A \times {}_A\mathcal{C}_A \rightarrow \mathcal{C}_A$  with  $A \otimes_A X \cong X$  in  ${}_A\mathcal{C}$  and  $Y \otimes_A A \cong Y$  in  $\mathcal{C}_A$ .



We can also construct  ${}_A[-, -]: {}_A\mathcal{C} \times {}_A\mathcal{C} \rightarrow \mathcal{C}_0$  as an equalizer

$${}_A[X, X'] \longrightarrow [X, X'] \rightrightarrows [A \otimes_1 X, X']$$

and similarly for  $[-, -]_A: \mathcal{C}_A \times \mathcal{C}_A \rightarrow \mathcal{C}_0$ . Again we get induced functors  ${}_A[-, -]: {}_A\mathcal{C} \times {}_A\mathcal{C}_A \rightarrow \mathcal{C}_A$  and  $[-, -]_A: \mathcal{C}_A \times {}_A\mathcal{C}_A \rightarrow {}_A\mathcal{C}$ .

There are natural isomorphisms:

$$\begin{aligned} {}_A\mathcal{C}(X \otimes_1 Z, X') &\cong \mathcal{C}_0(Z, {}_A[X, X']), \\ \mathcal{C}_A(Z \otimes_1 Y, Y') &\cong \mathcal{C}_0(Z, [Y, Y']_A), \text{ and} \\ {}_A[X \otimes_1 Z, X'] &\cong [Z, {}_A[X, X']]. \end{aligned}$$

That is,  $X \otimes_1 -$  and  ${}_A[X, -]$  (and  $- \otimes_1 Y$  and  $[Y, -]_A$ ) are adjoints. We denote the unit and counit of the adjunctions by  $\eta$  and  $\varepsilon$ . As  $A \in {}_A\mathcal{C}_A$ , we can define functors  $(-)^* = {}_A[-, A]: {}_A\mathcal{C} \rightarrow \mathcal{C}_A$  and  $(-)^* = [-, A]_A: \mathcal{C}_A \rightarrow {}_A\mathcal{C}$  with  $A^* \cong A$ , where on one side,  $A$  is regarded as a left  $A$ -module, and on the other side,  $A$  is regarded as a right  $A$ -module. Also notice that  ${}_1\mathcal{C} \cong \mathcal{C}_0 \cong \mathcal{C}_1$ . Again all the details are in [15].

In accordance with our convention from Setup 1, we often write  $\mathcal{C}$  for either  ${}_A\mathcal{C}$  or  $\mathcal{C}_A$ .

The forgetful functor from  ${}_A\mathcal{C} \rightarrow \mathcal{C}_0$  creates limits, colimits and isomorphisms [15, 2.4] and thus we get:

**Proposition 4.** *If  $\mathcal{C}_0$  is abelian, then so is  ${}_A\mathcal{C}$ . If  $\mathcal{C}_0$  is generated by a collection of objects  $\{X\}$ , then  ${}_A\mathcal{C}$  is generated by the collection  $\{A \otimes_1 X\}$ . If  $\mathcal{C}_0$  is Grothendieck, then  ${}_A\mathcal{C}$  is Grothendieck.*

*Proof.* We only prove the assertion about the generators (see Definition 2). Assume that  $\mathcal{C}_0$  is generated by  $\{X\}$ . Let  $Y \rightarrow Y'$  be a non-zero morphism in  ${}_A\mathcal{C}$ . Then  $Y \rightarrow Y'$  is also non-zero in  $\mathcal{C}_0$ , so we can find some  $X$  in the collection  $\{X\}$  and a morphism  $f: X \rightarrow Y$  such that  $X \rightarrow Y \rightarrow Y'$  is non-zero in  $\mathcal{C}_0$ . Now the morphism  $X \rightarrow A \otimes_1 X \rightarrow A \otimes_1 Y \rightarrow Y \rightarrow Y'$  is non-zero as  $X \rightarrow A \otimes_1 X \rightarrow A \otimes_1 Y \rightarrow Y$  is equal to  $f$ , and hence  $A \otimes_1 X \rightarrow A \otimes_1 Y \rightarrow Y \rightarrow Y'$  must be non-zero as well. Thus the collection  $\{A \otimes_1 X\}$  generates  ${}_A\mathcal{C}$ .  $\square$

**4.1. Dualizable objects.** In [12, III§1] Lewis and May define *finite* objects in a closed symmetric monoidal category. Such objects are called (*strongly*) *dualizable* in Hovey, Palmieri, and Strickland [10]. We extend this notion to categories of left/right modules over a monoid in a closed symmetric monoidal category by the following definition.

**Definition 7.** We call an object  $X \in {}_A\mathcal{C}$  *dualizable* if the counit  $\varepsilon: X^* \otimes_1 X \rightarrow A$  induces an isomorphism

$$\mathcal{C}_0(1, X^* \otimes_A X) \rightarrow {}_A\mathcal{C}(X, X);$$

the induced map sends an  $f$  to  $X \cong X \otimes_1 1 \xrightarrow{X \otimes_1 f} X \otimes_1 X^* \otimes_A X \xrightarrow{\varepsilon \otimes_A X} A \otimes_A X \cong X$ . Similarly one defines what it means for an object in  $\mathcal{C}_A$  or in  $\mathcal{C}_0$  to be dualizable.

Note that  $A \in {}_A\mathcal{C}$  and  $A \in \mathcal{C}_A$  are always dualizable.

For  $X, X' \in {}_A\mathcal{C}$  and  $Z \in {}_A\mathcal{C}_A$  the counit  $\varepsilon$  also induces a morphism

$$X \otimes_1 {}_A[X, Z] \otimes_A X' \xrightarrow{\varepsilon \otimes_A X'} Z \otimes_A X',$$

in  ${}_A\mathcal{C}$  which by adjunction yields a morphism  $\nu: {}_A[X, Z] \otimes_A X' \rightarrow {}_A[X, Z \otimes_A X']$  in  $\mathcal{C}_0$ . In the special case  $X' = X$  and  $Z = A$  we get  $\nu: X^* \otimes_A X \rightarrow {}_A[X, X]$ . These morphisms are used in the following characterization of dualizable objects.

**Lemma 5.** *The following conditions are equivalent for  $X \in {}_A\mathcal{C}$ .*

- (1)  $X$  is dualizable.
- (2)  $\varepsilon$  induces an isomorphism in any one of the following cases:
  - (a)  $\mathcal{C}_0(1, X^* \otimes_A -) \rightarrow {}_A\mathcal{C}(X, -)$  as functors on  ${}_A\mathcal{C}$ .
  - (b)  $\mathcal{C}_0(Z, X^* \otimes_A X') \rightarrow {}_A\mathcal{C}(X \otimes_1 Z, X')$  for any  $X' \in {}_A\mathcal{C}$  and  $Z \in \mathcal{C}_0$ .
- (3)  $\nu$  induces an isomorphism in any one of the following cases:
  - (a)  $X^* \otimes_A X \cong {}_A[X, X]$
  - (b)  ${}_A[X, Z] \otimes_A X' \cong {}_A[X, Z \otimes_A X']$  for any  $X' \in {}_A\mathcal{C}$  and  $Z \in {}_A\mathcal{C}_A$ .
  - (c)  ${}_A[X', Z] \otimes_A X \cong {}_A[X', Z \otimes_A X]$  for any  $X' \in {}_A\mathcal{C}$  and  $Z \in {}_A\mathcal{C}_A$ .
- (4) There exists a morphism  $1 \rightarrow X^* \otimes_A X$  making the next diagram commute:

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & X^* \otimes_A X \\ \eta \downarrow & \swarrow \nu & \\ {}_A[X, X] & & \end{array}$$

- (5)  $X^* \otimes_A -$  has left adjoint  $X \otimes_1 -$ , i.e.  $\mathcal{C}_0(Z, X^* \otimes_A X') \cong {}_A\mathcal{C}(X \otimes_1 Z, X')$  naturally for any  $X' \in {}_A\mathcal{C}$  and  $Z \in \mathcal{C}_0$ . In particular, the functor  $X^* \otimes_A -$  is exact.
- (6) There is some  $Y \in \mathcal{C}_A$  such that  $\mathcal{C}_0(Z, Y \otimes_A X') \cong {}_A\mathcal{C}(X \otimes_1 Z, X')$  naturally for any  $X' \in {}_A\mathcal{C}$  and  $Z \in \mathcal{C}_0$ .

*Proof.* (4) is equivalent to (1) just by looking at the adjoint diagram and unfolding the definition and to (2) by invoking the Yoneda lemma. Clearly (2.b) implies (5) and (6), and either of (3.c) and (3.b) implies (3.a) which implies (4). That (5) and (6) implies (4) can be proved as in [12, Thm 1.6] and that (4) implies (3.c) and (3.b) as in [12, Prop. 1.3(ii)]. These proofs are for symmetric monoidal categories but they carry straight over.  $\square$

*Remark 4.* We notice that Lemma 5 (6) makes no mention of the functor  $[-, -]$  and thus this condition can be used to define dualizable objects in, for example, symmetric monoidal categories that are not closed. In this case,  $Y$  is a “dual” of  $X$ . We chose a definition with a fixed dual object,  $X^* = {}_A[X, A]$ , because this emphasizes the canonical and thereby functorial choice of a dual object.

Next we show three lemmas about closure properties for the class of dualizable objects.

**Lemma 6.**  $(-)^*$  induces a duality between the categories of dualizable objects in  ${}_A\mathcal{C}$  and dualizable objects in  $\mathcal{C}_A$ . In particular, if  $X$  is dualizable, then so is  $X^*$  and the adjoint of  $\varepsilon$  gives an isomorphism  $X \rightarrow X^{**}$ .

*Proof.* As in [12, Prop. 1.3(i)].  $\square$

**Lemma 7.** Dualizable objects are closed under extensions and direct summands.

*Proof.* The closure under direct summands follows directly from Lemma 5 (2a).

So assume that

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$$

is exact and  $X_1, X_3$  are dualizable (in  ${}_A\mathcal{C}$ ). Then we have the following commutative diagram in  $\mathcal{C}_0$  with exact rows

$$\begin{array}{ccccccc} X_2^* \otimes_A X_1 & \longrightarrow & X_2^* \otimes_A X_2 & \longrightarrow & X_2^* \otimes_A X_3 & \longrightarrow & 0 \\ \simeq \downarrow & & \downarrow & & \simeq \downarrow & & \\ 0 \longrightarrow & {}_A[X_2, X_1] & \longrightarrow & {}_A[X_2, X_2] & \longrightarrow & {}_A[X_2, X_3], & \end{array}$$

where the outer vertical morphisms are isomorphisms by Lemma 5 (3.c), so the middle morphism is an isomorphism by the snake lemma. Hence  $X_2$  is dualizable by Lemma 5 (3.a).  $\square$

**Lemma 8.** *If  $S$  is dualizable in  $\mathcal{C}_0$ , then*

$$(X \otimes_1 S)^* \cong S^* \otimes_1 X^*$$

*for any  $X \in {}_A\mathcal{C}$ . If  $X \in {}_A\mathcal{C}$  is dualizable then so is  $X \otimes_1 S \in {}_A\mathcal{C}$ . In particular,  $A \otimes_1 S \in {}_A\mathcal{C}$  and  $(A \otimes_1 S)^* \cong S^* \otimes_1 A \in \mathcal{C}_A$  are dualizable if  $S \in \mathcal{C}_0$  is dualizable.*

*Proof.* If  $S \in \mathcal{C}_0$  is dualizable, then we have

$$(X \otimes_1 S)^* = {}_A[X \otimes_1 S, A] \cong [S, {}_A[X, A]] \cong [S, 1 \otimes_1 X^*] \cong [S, 1] \otimes_1 X^* = S^* \otimes_1 X^*.$$

When  $X$  is also dualizable we have

$$\mathcal{C}_0(1, (X \otimes_1 S)^* \otimes_A -) \cong \mathcal{C}_0(1, S^* \otimes_1 X^* \otimes_A -) \cong \mathcal{C}_0(S, X^* \otimes_A -) \cong {}_A\mathcal{C}(X \otimes_1 S, -)$$

on  ${}_A\mathcal{C}$ , and hence  $X \otimes_1 S$  is dualizable in  ${}_A\mathcal{C}$  by Lemma 5 (2).  $\square$

We now have a large supply of categories satisfying Setup 1.

**Proposition 5.** *Let  $A$  be a monoid in a closed symmetric monoidal Grothendieck category  $(\mathcal{C}_0, \otimes_1, [-, -], 1)$  where  $1$  is finitely presented. Assume that  $\mathcal{C}_0$  is generated by a class  $\mathcal{S}$  of dualizable objects such that  $\mathcal{S}^*$  also generates  $\mathcal{C}_0$  (e.g. if  $\mathcal{S} = \mathcal{S}^*$ ). Assume furthermore that  ${}_A\mathcal{S}$  is a class of dualizable objects in  ${}_A\mathcal{C}$  which is closed under extensions and contains  $A \otimes_1 \mathcal{S}$  (e.g.  ${}_A\mathcal{S}$  could be the class of all dualizable objects in  ${}_A\mathcal{C}$ ; see Lemmas 7 and 8).*

*Then  ${}_A\mathcal{C}$  and  ${}_A\mathcal{S}$  satisfy Setup 1 (more precisely,  $\mathcal{C}_L := {}_A\mathcal{C}$ ,  $\mathcal{C}_R := \mathcal{C}_A$ ,  $\otimes := \otimes_A$ ,  $(-)^* := {}_A[-, A]$ ,  $\mathcal{S}_L := {}_A\mathcal{S}$  and  $\mathcal{S}_R := ({}_A\mathcal{S})^*$ ).*

*Proof.* As each of the classes  $\mathcal{S}$  and  $\mathcal{S}^*$  generate  $\mathcal{C}_0$ , the class  $A \otimes_1 \mathcal{S}$  generates  ${}_A\mathcal{C}$  and  $\mathcal{S}^* \otimes_1 A$  generates  $\mathcal{C}_A$  by Proposition 4. By assumption,  $A \otimes_1 \mathcal{S}$  is contained in  ${}_A\mathcal{S}$ , so in view of Lemma 8,  $\mathcal{S}^* \otimes_1 A = (A \otimes_1 \mathcal{S})^*$  is contained in  $({}_A\mathcal{S})^*$ . Hence  ${}_A\mathcal{S}$  and  $({}_A\mathcal{S})^*$  generate  ${}_A\mathcal{C}$  and  $\mathcal{C}_A$ . By Lemma 6 the class  $({}_A\mathcal{S})^*$  consists of dualizable objects and  $(-)^*$  yields the desired duality. Since  ${}_A\mathcal{S}$  is closed under extensions, it follows that the same is true for  $({}_A\mathcal{S})^*$ . The natural isomorphisms in Setup 1 hold by Lemma 5 (2.a). As dualizable objects are, in particular, finitely presented (see Remark 2), it follows from Proposition 4 and from Proposition 1 (1) that  $\mathcal{C}_0$ ,  ${}_A\mathcal{C}$ , and  $\mathcal{C}_A$  are locally finitely presented Grothendieck categories. It remains to note that  $\otimes_A$  is a right continuous bifunctor.  $\square$

We now introduce a condition (which in some cases can be checked, see Section 5) which guarantees that Setup 2 will hold as well.

**Lemma 9.** *Consider the situation from Proposition 5. If  $1 \in \mathcal{C}_0$  has an augmented projective resolution consisting of finitely presented objects, that stays exact under  $- \otimes_1 S$  for any  $S \in \mathcal{C}_0$  (or just for any  $S \in {}_A\mathcal{S}$ ), then Setup 2 holds.*

*Proof.* Let  $P_\bullet$  be the assumed (non-augmented) projective resolution of 1. Then

$$\mathrm{Ext}_{\mathcal{C}_0}(1, \varinjlim X_i) \cong H^1\mathcal{C}_0(P_\bullet, \varinjlim X_i) = \varinjlim H^1\mathcal{C}_0(P_\bullet, X_i) = \varinjlim \mathrm{Ext}_{\mathcal{C}_0}(1, X_i),$$

so 1 is  $\mathrm{FP}_2$ . Here we have used that each  $P_i$  is finitely presented and that cohomology  $H^1$  respects direct limits. Furthermore, every  $S \in \mathcal{S}_L = {}_A\mathcal{S}$  is dualizable and hence the functor  $-\otimes_A S$  is exact by (the dual of) Lemma 5 (5). It remains to show the two natural isomorphisms from Setup 2.

For this let  $S \in {}_A\mathcal{S}$  and notice that  $P_\bullet \otimes_1 S^*$  is a projective resolution of  $S^*$ . Indeed, it is exact by assumption and it consists of projectives as  $-\otimes_1 S^*$  has right adjoint  $-\otimes_A S$  which is exact. We can now calculate:

$$\mathrm{Ext}_{\mathcal{C}_0}(1, -\otimes_A S) \cong H^1\mathcal{C}(P_\bullet, -\otimes_A S) \cong H^1\mathcal{C}_A(P_\bullet \otimes_1 S^*, -) \cong \mathrm{Ext}_{\mathcal{C}_A}(S^*, -).$$

The other of the required natural isomorphisms from Setup 2 is proved similarly.  $\square$

*Remark 5.* Consider the situation from Proposition 5.

- If 1 is projective, then as in the proof above any  $S \in \mathcal{S}$  is projective, so in the cotorsion pair  $(\mathcal{P}, \mathcal{E})$  generated by  $\mathcal{S}$ , the class  $\mathcal{P}$  consists of all projective  $A$ -modules and  $\mathcal{E}$  consists of all  $A$ -modules.
- In the case  $A = 1$  the hypothesis of Lemma 9 is satisfied if 1 has a projective resolution of finitely presented objects. Indeed, it is automatic that such a resolution stays exact under the functor  $-\otimes_1 S$  when  $S \in \mathcal{C}_0$  is dualizable.
- If the projective resolution of 1 consist of dualizable objects, then as in the proof above every dualizable object in  $\mathcal{S}$  has such a resolution, so as in the proof of Theorem 1 every semi-flat  $M \in {}_A\mathcal{C}$  satisfies  $\mathrm{Ext}_{\mathcal{C}_0}(S, E \otimes_A M) = 0$  for any dualizable  $S$  in  $\mathcal{C}_0$  and any acyclic  $E$  in  $\mathcal{C}_A$ . That is,  $\mathcal{S}_A^\perp \otimes_A M$  is right orthogonal to all dualizable objects in  $\mathcal{C}_0$ , not just to  $1 \in \mathcal{C}_0$ .

## 5. EXAMPLES

In this final section, we return to the examples from Example 1 and to the results from the literature mentioned in the Introduction.

**5.1.  $A\text{-Mod}$ .**  $\mathcal{C}_0 = \mathrm{Ab}$  is a locally finitely presented abelian category generated by  $1 = \mathbb{Z}$  which is projective, and  ${}_A\mathcal{C}$  is just  $A\text{-Mod}$ . The condition of Lemma 5 (4) is equivalent to the existence of a finite number of elements  $f_i \in X^*$  and  $x_i \in X$  such that  $x = \sum_i f_i(x)x_i$  for any  $x \in X$ . By the Dual Basis Theorem [13, Chap. 2.3], this is precisely the finitely generated projective  $R$ -modules. Also the finitely generated free modules are closed under extensions and contains  $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ , so by Corollary 1, Proposition 5 and Remark 5 we get the original theorem of Lazard and Govorov:

**Corollary 2.** *Over any ring, the flat modules are the direct limit closure of the finitely projective (or free) modules.*

**5.2.  $A\text{-DGMod}$ .** Let  $\mathcal{C}_0 = \mathrm{Ch}(\mathrm{Ab})$  be the category of chain complexes of abelian groups, where 1 is the complex with  $\mathbb{Z}$  concentrated in degree 0. This is finitely presented, as  $\mathcal{C}_0(1, -) \cong Z_0(-)$  is the 0<sup>th</sup> cycle functor which preserves direct limits. The category is locally finitely presented abelian, generated by shifts of  $M(\mathrm{Id}_1)$ . Here  $M(\mathrm{Id}_1)$  is the mapping cone of the identity morphism on 1; this complex has  $\mathbb{Z}$  in degrees 1 and 0 and zero otherwise, the differential being the identity on  $\mathbb{Z}$ . A monoid  $A$  in  $\mathrm{Ch}(\mathrm{Ab})$  is simply a differential graded algebra and  ${}_A\mathcal{C}$  is the category  $A\text{-DGMod}$  of differential graded left  $A$ -modules. DG-modules are thus covered by

Setup 3. Clearly any shift of  $A$  is dualizable, so by Lemma 7 any finite extension of shifts of  $A$  will be dualizable, and we call such modules *finitely generated semi-free*. Direct summands of those are called *finitely generated semi-projectives*. It is not hard to check that both classes are self-dual and closed under extensions. They also contain  $A \otimes_{\mathbb{Z}} \Sigma^i M(\text{Id}_1)$  for any  $i$ , thus satisfy Setup 1 by Proposition 5.

We also have

**Lemma 10.** *In  $\text{Ch}(\text{Ab})$  the object  $1 = \mathbb{Z}$  (viewed as a complex concentrated in degree 0) has a projective resolution of finitely presented (in fact dualizable) objects that stays exact when tensored with any object from  $\text{Ch}(\text{Ab})$ .*

*Proof.* This we can construct using the mapping cone. The mapping cone  $M(\text{Id}_1)$  is both projective and dualizable, and we thus have a projective resolution of finitely presented objects

$$\cdots \longrightarrow \Sigma^3 M(\text{Id}_1) \longrightarrow \Sigma^2 M(\text{Id}_1) \longrightarrow \Sigma M(\text{Id}_1) \longrightarrow 1 \longrightarrow 0,$$

which after applying  $- \otimes_{\mathbb{Z}} X$  for  $X \in \text{Ch}(\text{Ab})$  becomes

$$\cdots \longrightarrow \Sigma^3 M(\text{Id}_X) \longrightarrow \Sigma^2 M(\text{Id}_X) \longrightarrow \Sigma M(\text{Id}_X) \longrightarrow X \longrightarrow 0,$$

which is still exact.  $\square$

Thus, by Lemmas 9 and 10 we get that DG-modules together with the finitely generated semi-free or finitely generated projective objects satisfy Setup 2.

Now before stating the corollary to our main theorem, let's see that our abstract notions of semi-projective, acyclic and semi-flat objects from Definition 6 agree with the usual ones. These notions originate in the treatise [1] by Avramov, Foxby, and Halperin, where several equivalent conditions are given.

**Definition 8.** Let  $A$  be any DGA and let  ${}_A\mathcal{C} = A\text{-DGMod}$ .

- A DG-module is called *acyclic* (or *exact*) if it has trivial homology.
- A DG-module,  $P$ , is called *semi-projective* (or *DG-projective*) if  ${}_A\mathcal{C}(P, \psi)$  is epi, whenever  $\psi$  is epi and  $\ker \psi$  has trivial homology (in other words,  $\psi$  is a surjective quasi-isomorphism).
- A DG-module,  $M$ , is called *semi-flat* (or *DG-flat*) if  $- \otimes_A M$  is exact and preserves acyclicity (i.e.  $E \otimes_A M$  has trivial homology whenever  $E$  has).

First we notice that:

**Lemma 11.** *A DG-module  $P$  is DG-projective iff  $\text{Ext}_{{}_A\mathcal{C}}^1(P, E) = 0$  whenever  $E$  is a DG-module with trivial homology.*

*Proof.* If  $\text{Ext}_{{}_A\mathcal{C}}^1(P, E) = 0$  and

$$0 \longrightarrow E \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$$

is an exact sequence, then clearly  ${}_A\mathcal{C}(P, \varphi)$  is epi. On the other hand, if

$$0 \longrightarrow E \longrightarrow X \xrightarrow{\varphi} P \longrightarrow 0$$

is exact and  ${}_A\mathcal{C}(P, \varphi)$  is epi, then the sequence split, so  $\text{Ext}_{{}_A\mathcal{C}}^1(P, E) = 0$ .  $\square$

Next we see that:

**Lemma 12.** *Let  $A$  be a DGA. For any  $N \in {}_A\mathcal{C}$  we have  $\text{Ext}_{{}_A\mathcal{C}}^1(\Sigma A, N) = H^0(N)$ .*

*Proof.* To compute this, we use the short exact sequence

$$0 \longrightarrow A \longrightarrow M(\mathrm{Id}_A) \longrightarrow \Sigma A \longrightarrow 0$$

where  $M(\mathrm{Id}_A)$  is the mapping cone of  $A \xrightarrow{\mathrm{Id}_A} A$ . Since  $M(\mathrm{Id}_A)$  is projective we have  $\mathrm{Ext}_{A\mathcal{C}}^1(M(\mathrm{Id}_A), N) = 0$ , so we get an exact sequence

$${}_A\mathcal{C}(M(\mathrm{Id}_A), N) \longrightarrow {}_A\mathcal{C}(A, N) \longrightarrow \mathrm{Ext}_{A\mathcal{C}}^1(\Sigma A, N) \longrightarrow 0$$

Straightforward calculations show that this sequence is isomorphic to

$$N^{-1} \xrightarrow{\partial_N^{-1}} Z^0(N) \longrightarrow \mathrm{Ext}_{A\mathcal{C}}^1(\Sigma A, N) \longrightarrow 0$$

where  $N^{-1}$  is the (cohomological) degree  $-1$  part of  $N$  and  $\partial_N^{-1}$  is the differential. Thus we get the desired isomorphism  $\mathrm{Ext}_{A\mathcal{C}}^1(\Sigma A, N) \cong H^0(N)$ .  $\square$

Together we have:

**Theorem 2.** *Let  $\mathcal{S}$  be the class of finitely generated semi-free/semi-projective DG-modules over any DG-algebra  $A$  (see 5.2). In the category of DG  $A$ -modules, the cotorsion pair generated by  $\mathcal{S}$  is complete and it is given by*

$$(DG\text{-projective DG-modules, exact DG-modules}).$$

*The direct limit closure of  $\mathcal{S}$  is the class of semi-flat (or DG-flat) DG-modules.*

*Proof.* Let  $\mathcal{P}$  be the class of DG-projective DG-modules, and  $\mathcal{E}$  the class of exact DG-modules (i.e. with trivial homology). From Lemma 12 (and by using shift  $\Sigma$ ) we have  $\mathcal{S}^\perp \subseteq \mathcal{E}$ , and from Lemma 11 we have  $\mathcal{P} = {}^\perp\mathcal{E}$ . Now since  $\mathcal{S} \subseteq \mathcal{P}$  we have  $\mathcal{E} \subseteq ({}^\perp\mathcal{E})^\perp = \mathcal{P}^\perp \subseteq \mathcal{S}^\perp$ , and hence  $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp) = (\mathcal{P}, \mathcal{E})$ . Completeness of the cotorsion pair follows from Proposition 2 since  $\mathcal{S}$  is skeletally small by Proposition 1 (4). As in Remark 5 the abstract notion of semi-flatness agrees with the DG-notion, so the last statement follows from Theorem 1.  $\square$

*Remark 6.* Without Setup 2 we can directly see that the abstract notion and the DG-notion of preservation of acyclicity agree (Definitions 6 and 8), using Lemma 12 and the fact that  $- \otimes_A M$  preserves shifts. That any  $S \in \varinjlim_A \mathcal{S}$  is semi-flat also follows from results in [1], where it is proved that any semi-projective is semi-flat and the semi-flat are closed under direct limit. The cotorsion pair has already been studied, at least in the case where the DGA is just a ring (and hence DG-modules are just complexes), though I could not find a reference for DG-modules over a general DGA.

**5.3.  $\mathbf{Ch}(A)$ .** In the case of complexes over a ring  $A$  a direct calculation using the dual basis theorem component-wise, shows that the dualizable objects in  $\mathbf{Ch}(A)$  are precisely the perfect complexes. From above we thus have:

**Corollary 3.** *Let  $A$  be any ring and let  $\mathcal{S}$  be the class of perfect  $A$ -complexes. In the category  $\mathbf{Ch}(A)$ , the cotorsion pair generated by  $\mathcal{S}$  is complete and it is given by (semi-projective complexes, acyclic complexes). The direct limit closure of  $\mathcal{S}$  is the class of semi-flat complexes.*

*Remark 7.* This cotorsion pair has already been studied for instance in [7] where 2.3.5 and 2.3.6 proves it is a cotorsion pair, and 2.3.25 that it is complete (with slightly different notation). It is not mentioned, however, that it is generated by a

set. As already mentioned the direct limit closure has in this case been worked out in [4].

**5.4.  $\mathbf{QCoh}(\mathbf{X})$ .** Let  $X$  be a scheme and let  $\mathbf{QCoh}(X)$  be the category of quasi-coherent sheaves on  $X$ . The *dualizable* objects are also studied in [2, 4.7.1 and 4.7.2], and [2, 4.7.5] shows that they are exactly the locally free sheaves of finite rank. Thus, from Theorem 1 and Example 1 (4) we get:

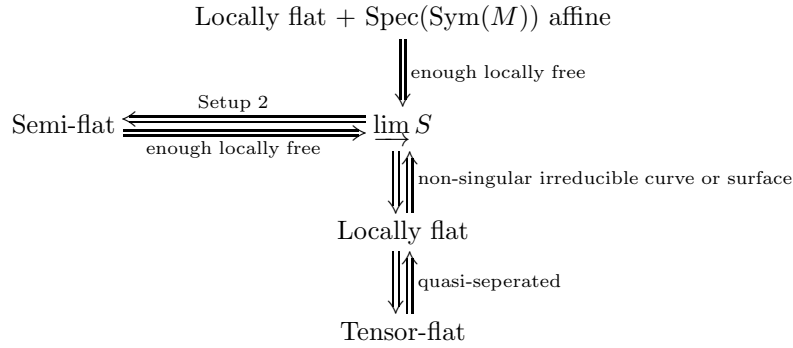
**Proposition 6.** *Let  $X$  be a scheme such that  $\mathcal{C} = \mathbf{QCoh}(X)$  has enough locally free, i.e.  $\mathcal{C}$  is generated by the class  $\mathcal{S}$  of locally free sheaves of finite rank. If  $M$  is semi-flat, then  $M \in \varinjlim \mathcal{S}$ . If  $\mathcal{C}$  and  $\mathcal{S}$  satisfy Setup 2 then any  $M \in \varinjlim \mathcal{S}$  is semi-flat.*

This is not the first Lazard-like theorem for quasi-coherent sheaves.

In [5, (5.4)] Crawley-Boevey proves that  $\varinjlim \mathcal{S}$  is the locally flat sheaves if  $X$  is a non-singular irreducible curve or surface over a field  $k$ .

In [2, 2.2.4] Brandenburg proves that if  $\mathbf{QCoh}(X)$  has enough locally free and  $M$  is locally flat and  $\mathrm{Spec}(\mathrm{Sym}(M))$  is affine, then  $M \in \varinjlim \mathcal{S}$ .

These results give us the following relations



I'm sure much more can be said about this.

In particular it could be good to get rid of Setup 2 as I don't know when this is satisfied for sheaves, as  $\mathbf{QCoh}(X)$  has no non-trivial projectives in general.

It would also be interesting to get a concrete description of the semi-flat, the acyclic and the semi-projectives.

Some work has been done in this direction. In Enochs, Estrada, and García-Rozas [6, 3.1] we see that the semi-projective objects are locally projective, and in [6, 4.2] we see that they are precisely the locally projective sheaves in the special case of  $P_1(k)$  (the projective line over an algebraically closed field  $k$ ). In this case a concrete computational description of the (abstract) acyclic objects are given which is equivalent to the vanishing of (higher) cohomology. I am not aware of anybody explicitly studying semi-flat sheaves.

**5.5. Additive functors.** Following [14], let  $\mathcal{C}_0 = \mathbf{Ab}$ , let  $\mathcal{X}$  be a preadditive category, let  $\mathcal{X}^{\mathrm{op}}$  be the dual category and let  $\mathcal{C}_L = [\mathcal{X}, \mathbf{Ab}]$  and  $\mathcal{C}_R = [\mathcal{X}^{\mathrm{op}}, \mathbf{Ab}]$  be the categories of additive functors. These are abelian and locally finitely presented by [14, Lem. 2.4]. As in [14] one can define a tensorproduct

$$\otimes_{\mathcal{X}}: [\mathcal{X}^{\mathrm{op}}, \mathbf{Ab}] \times [\mathcal{X}, \mathbf{Ab}] \rightarrow \mathbf{Ab}.$$

We call functors  $\mathcal{X}(-, x)$  and  $\mathcal{X}(x, -)$  representable, where  $x \in \mathcal{X}$ , and define  $\mathcal{X}(-, x)^* = \mathcal{X}(x, -)$  and vice versa. As asserted in Example 1 (5), these categories together with  $\mathcal{S} = \{\text{finite direct sums of representable functors}\}$  satisfy Setup 1.

This is because first of all  $\mathcal{S}$  is closed under extensions. Indeed, the objects in  $\mathcal{S}$  are projective (in fact, every projective object is a direct summand of a direct sum of representable functors), hence any extension is a direct sum. If  $\mathcal{X}$  is additive, then  $\mathcal{X}(-, x) \oplus \mathcal{X}(-, y) \cong \mathcal{X}(-, x \oplus y)$ , so in this case  $\mathcal{S}$  is just the class of representable functors (finite direct sums are not needed).

Further, as in [14] the tensor product is such that for any  $F \in \mathcal{C}_L$  and  $G \in \mathcal{C}_R$  we have

$$\mathcal{X}(-, x) \otimes_{\mathcal{X}} F \cong Fx \text{ and } G \otimes_{\mathcal{X}} \mathcal{X}(x, -) \cong Gx.$$

which by the Yoneda lemma, and the fact that  $\text{Ab}(1, -)$  is the identity gives the required isomorphisms.

Since  $1 = \mathbb{Z} \in \text{Ab}$  is projective, Corollary 1 gives a new proof of [14, Thm 3.2]:

**Corollary 4.** *Let  $\mathcal{X}$  be an additive category, and let  $\mathcal{S}$  be the finitely generated projective functors or the representable functors in  $[\mathcal{X}, \text{Ab}]$  (or the direct sums of representable functors if  $\mathcal{X}$  is only preadditive). A functor  $F$  is flat iff  $F \in \varinjlim \mathcal{S}$ .*

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